Vol 8, Issue 3, 2025 **Impact Factor: 8.535** DOI: https://doi.org/10.31426/ijamsr.2025.8.3.8135

International Journal of
Advanced Multidisciplinary Scientific Research (IJAMSR) ISSN:2581-4281

Efficient Solution of Nonlinear Delay Differential Equations Via Differential Transform Method Combined with Bell Polynomials

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ABSTRACT

Combining the Differential Transform Method (DTM) with Bell polynomials, this work offers an effective hybrid method for resolving DDEs that are not linear in time. Using the characteristics of Bell polynomials, the approach successfully solves a system of p-dimensional proportional delay differential equations with nonlinear components. The suggested method uses differential transforms to change the initial conditions and DDEs provided, and then uses Bell polynomial representations to extend the nonlinear components. Next, we build a recursive system to find the transform coefficients that are unknown. The last step is to use the inverse differential transform to get a quickly convergent series that represents an approximation of the analytical answer. In order to prove the effectiveness and correctness of the technique, it is evaluated on three cases. The hybrid technique outperforms the state-of-the-art methods in terms of accuracy and reliability, according to comparison results that include absolute and maximum error calculations. Because it streamlines dealing with delays and nonlinearities, this method is useful for solving complicated differential systems.

Keywords: *Differential Transform Method, Bell Polynomials, Delay Differential Equations, Nonlinear Systems.*

I. INTRODUCTION

The development of a system is affected by both its current and past states in delay differential equations (DDEs), a crucial type of functional differential equations. Engineering, control theory, economics, epidemiology, and population dynamics are just a few of the many areas that make extensive use of these equations. The already difficult equations known as nonlinear delay differential equations (NDDEs) get even more so when nonlinearity and delays are both included. Because of the challenges of concurrently managing the nonlinear components and the delay terms, traditional analytical approaches frequently fail. Therefore, finding computationally easy, efficient, and accurate ways to solve NDDEs is an important and ongoing research topic.

The (DTM) is one of the most effective numerical and semi-analytical techniques to NDDEs that has been devised. Using a fast-convergent series as an approximation, DTM offers an effective approach for solving differential equations. It is based on the Taylor series expansion. This technique simplifies the initial differential equations by converting them into a system of recurrence relations that may be solved repeatedly. In contrast to completely numerical approaches, DTM keeps the solution's analytical framework, which makes it great for sensitivity analysis and investigating qualitative behavior.

It can be more challenging to apply DTM to nonlinear concepts, particularly ones incorporating delayed arguments. Complex expressions arising from nonlinearities are typically beyond the capabilities of the DTM framework. By including Bell polynomials, a substantial improvement is made to overcome this constraint. Nonlinear composite functions, including products and higher-order derivatives, can be compactly and computationally expressed using Bell polynomials. With the addition of Bell polynomials to the DTM framework, nonlinear terms may be handled more effectively, leading to the generation of explicit recursive formulae that are both simple and accurate, much like the original technique.

One strong method for solving NDDEs is the combination approach, which consists of DTM plus Bell polynomials. Simple implementation and fast convergence are made possible by this hybrid technique, which helps to decompose complicated nonlinear delay terms into computable components. This method is superior to perturbation approaches in that it does not necessitate tiny parameters. Additionally, it is superior to purely numerical schemes in that it avoids the complications associated with numerical instability and truncation errors caused by discretization and grid formation.

A large class of NDDEs, including those with constant, time-dependent, or state-dependent delays, may be implemented using this method. No matter the situation, the approach guarantees computational feasibility without compromising the solution's analytical nature. Because of this, it is ideal for use in simulations and modeling projects as well as in theoretical research.

In this research, we want to show that this combination approach works well for solving different kinds of delay differential equations that are not linear. By utilizing the method with examples, the study demonstrates how nonlinear and delayed components may be simplified by integrating Bell polynomials into the DTM framework. In order to ensure that the suggested approach is accurate and efficient, the acquired results are compared with precise or numerical answers. Thanks to its promising results, this method has the makings of a go-to resource for scientists and engineers working with complicated dynamical systems that exhibit delays and nonlinear interactions.

A notable void in the existing literature on NDDEs can be filled by creating more reliable analytical methods, such the DTM in conjunction with Bell polynomials. When dealing with complicated problems in real-world applications, this strategy offers an elegant, efficient, and powerful solution.

International Journal of
Advanced Multidisciplinary Scientific Research (IJAMSR) ISSN:2581-4281

In particular, for time-delay systems with nonlinear behavior, it is a useful addition to the existing toolbox. We establish the groundwork for future developments in the analytical and computational treatment of such systems with this work, which adds to the expanding corpus of knowledge on effective resolution of delay differential equations.

II. REVIEW OF LITERATURE

Janczkowski Fogaça, et al., (2024) In order to solve linear first-order ordinary differential equations, the Leibniz integrating factor offers a simple and reliable approach. Regrettably, when attempting to apply the approach to differential equations of second or higher order, the integrating components end up becoming partial differential equations that are reliant on both the independent and dependent variables. Even though there are a plethora of specialized ways to solve issues nowadays, many of them could be much simplified with an integrating factor that was only a function of the independent variable. Applied mathematics, engineering, and physics all rely on second and higher order ordinary differential equations, which is why they are so significant. This study introduces the generalized integrating factor for n-order linear ordinary differential equations. Curiously, the analytical homogeneous solutions and the analytical particular solutions can be generated separately in the proposed formulation. The Bessel, Cauchy-Euler, and constant coefficients cases are solved analytically by using numerical methods and examples from the literature. Particular emphasis is placed on the constant coefficients case because to its extensive use in mechanical and electrical engineering. As a result, we analyze and solve a class of excitation functions, including periodic and polynomial continuous excitation, the discontinuous Dirac's delta and the Heaviside step function, and our own. Our comparison of the analytical results with all existing methods—including indeterminate coefficients, parameter modifications, and the Laplace transform—confirms that this version of the Leibniz integrating factor is practical and accurate.

Altaie, Huda & Abdali et al., (2024) Solving first- and second-order differential equations that have emerged in physics and engineering is the primary focus of this position. Among these techniques are the VIM, ADM, DTM, and TAM, which stand for Variational Iteration and Decomposition, respectively. Notably, the results demonstrate that the VIM offers a series-form solution that converges to an exact answer. Except for the calculation of Adomain polynomials, ADM can be used directly without translation, even if DTM requires it. To solve nonlinear problems, TAM used simpler iterative algorithms that converge to the right solution without assumptions. The gathered findings rapidly approach the exact solution.

Ismael, Ruaa & Al-Fayadh et al., (2024) By repeatedly combining the homotopic perturbation method, the variational iteration method, and the Aboodh transform, this article aims to provide a method for solving many kinds of partial differential equations. The suggested method is proved to be both effective and feasible by applying it to coupled pseudo-parabolic equations, modified KdV, coupled KdV, and Korteweg-de Vries (KdV). It is demonstrated that the obtained accurate solutions of the KdV equations as individual equations and the connected pseudo-parabolic equations form a convergent series with components that may be easily calculated. It was shown to be a convergent series with easily calculable components. By sidestepping typical obstacles such as the difficult integration in the variation iteration approach, the transform convolution theorem, and difficulties in estimating the Lagrange multiplier, the suggested method makes solving KdV, wave-like, and pseudo-parabolic equations faster and easier.

Gürbüz, Burcu & Fernandez et al., (2024) Differential equations, both in theory and practice, have been crucial to the development of our knowledge of mathematics and launching new avenues of practical study. Various branches of mathematics, including qualitative theory of differential equations, have produced a plethora of novel ideas and approaches to solving analytical processes, systems of differential equations, ordinary and partial differential equations, and more. From a practical standpoint, the modeling of several dynamical systems or processes in the actual world relies on differential equations, which are essential. When it comes to researching differential equations and estimating their solutions, numerical methods are just as significant as analytical ones. This is because analytical procedures don't always succeed in capturing the complexity of a situation.

Uday, Venkatachalapathi et al., (2024) Differential equations provide a framework for modeling events that vary across space or time, and they are thus crucial to mathematical models of dynamic systems. Engineering, biology, economics, and the social sciences are just a few of the many fields that make use of these equations to describe the behavior of systems governed by change rates. This research examines the use of differential equations in dynamic system modeling, provides an overview of the many types of differential equations, discusses solutions, and weighs the pros and cons of employing them. Our sincere desire is that this discussion will highlight the fundamental role of differential equations in the analysis and forecasting of real-world system behavior.

III. DESCRIPTION OF THE METHOD

A set of p-differential equations describing delays is examined here.

$$w^{(n)}(v) = \sum_{j=0}^{J} a_j(v)\, w\left(\frac{v}{b_j}\right) + h(w) + m(v), \tag{1}$$

Where $w^{(n)}(v) = \left(w_1^{(n)}(v), \dots, w_p^{(n)}(v)\right)^T$, $w^{(n)} = \left(w_1(v), \dots, w_p(v)\right)^t$, m(v) $= \left(m_1(v), \dots, m_p(v)\right)^T$, and $h(w) = \left(h_1(w), \dots, h_p(w)\right)^T$ all have p dimensions, and the nonlinear function h (w) is given by

$$h(w) = \sum_{i=1}^{l} \beta_i f_i\left(g_i\left(\frac{v}{q_i}\right)\right)$$

With the initial conditions

$$\sum_{r=0}^{n-1} w^{(r)}(v_0) = w_\alpha, \alpha = 0, 1, \dots, n-1, \tag{2}$$

Vol 8, Issue 3, 2025 **Impact Factor: 8.535** DOI: https://doi.org/10.31426/ijamsr.2025.8.3.8135

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International Journal of Advanced Multidisciplinary Scientific Research (IJAMSR) ISSN:2581-4281

$\beta_i \in$R, l, J, $\in$N, $b_j \geq 1$, $q_i \geq 1$, a_j (.) Here a system of p-differential equations defining delays is studied. The procedure's steps are best characterized as

Step 1: Implement the differential transform to the initial conditions given in Equation (2).
Step 2: Apply the differential transform to Equation (1), and use Bell polynomials to handle the nonlinear terms present in the equation.
Step 3: Construct a system of recurrence relations by combining the results obtained from Steps 1 and 2.
Step 4: Solve the resulting recursive relations to compute the unknown coefficients, specifically W (0), W (1), and others as needed.
Step 5: Apply the inverse differential transform to the set of coefficients obtained, yielding the approximate series solution for the system defined by Equations (1) and (2).

IV. RESULTS AND DISCUSSION

As proof of the method's efficacy and dependability, we give three numerical examples of NDDEs. The MATHEMATICA program has been used for all computations.

As an initial example, think about the following differential equation for nonlinear delays:

$$w'''(v) = -2w'(v) + 8w\left(\frac{v}{2}\right)\sqrt{1 - w^2\left(\frac{v}{2}\right)} \qquad (3)$$

With initial conditions

$$w(0) = 1, w'(0) = 0, w''(0) = -4 \qquad (4)$$

Here is the precise answer:

$$w(v) = cos2v \qquad (5)$$

Table 1: Evaluation of Numerical Accuracy: w_{15}(v) vs. Exact Solution (Example 1)"

v	w(v)	wN(v)	RN(v)
0.1	0.98006658	0.98006658	0
0.2	0.92106099	0.92106099	1.2×10^{-16}
0.3	0.82533561	0.82533561	1.3×10^{-16}
0.4	0.69670671	0.69670671	1.9×10^{-15}
0.5	0.54030231	0.54030231	8.8×10^{-14}

Example 2. Consider the the nonlinear delay differential equation shown below

$$w''(v) = 2\ln\left(w\left(\frac{v}{3}\right)\right) + 9w^3\left(\frac{v}{3}\right) - 2v \qquad (6)$$

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International Journal of Advanced Multidisciplinary Scientific Research (IJAMSR) ISSN:2581-4281

Given the starting points

$w(0)=1, w'(0)=3$ (7)

The precise answer is provided as

$w(v)=e^{3v}$ (8)

Table 2: Accuracy Comparison: w_{10} (v) and Exact Solution for Example 2

v	w(v)	wN(v)	RN(v)
0.1	1.3498588	1.3498588	3.3×10^{-14}
0.2	1.8221188	1.8221188	5.2×10^{-11}
0.3	2.4596031	2.4596031	3.4×10^{-9}
0.4	3.3201169	3.3201167	6.2×10^{-8}
0.5	4.4816891	4.4816866	5.5×10^{-7}
0.6	6.0496475	6.0496286	3.1×10^{-6}
0.7	8.1661699	8.1660639	1.2×10^{-5}
0.8	11.023176	11.022702	4.3×10^{-5}
0.9	14.879732	14.877945	1.2×10^{-4}
1.0	20.085537	20.079665	2.9×10^{-4}

Example 3. This nonlinear delay differential equation should be considered.

$$(2v+1)w'(v)=2ve^{w\left(\frac{v}{2}\right)}+e^{w\left(\frac{v}{2}\right)}-2v^2-3v+1 \quad (9)$$

Given the starting points

W (0) = 0 (10)

Exact solution is given as

$w(v)=\log(2v+1)$ (11)

Table 3. Assessment of Numerical Approximation w_{15} (v) in Example 3

v	w(v)	wN(v)	RN(v)
0.1	0.18232156	0.18232156	1.8×10^{-12}
0.2	0.33647224	0.33647226	5.7×10^{-8}
0.3	0.47000363	0.4700149	2.3×10^{-5}
0.4	0.58778666	0.58879092	1.7×10^{-3}
0.5	0.69314718	0.72537185	4.6×10^{-2}

V. CONCLUSION

An innovative and effective approach to solving nonlinear delay differential equations with the help of the Differential Transform Method and Bell polynomials is introduced in this paper. The suggested method offers a methodical and precise solution to the problems caused by nonlinearity and delays by combining the best features of the two approaches. There are several benefits to using these series solutions instead of more conventional numerical and perturbative approaches, including their ease of computation and fast convergence. The method's accuracy and computing efficiency have been confirmed using sample problems. Modeling and analysis of delayed dynamical systems benefit greatly from its capacity to simplify complicated nonlinear expressions while preserving the analytical quality of solutions. Applying this method to systems with stochastic components or variable delays is an area where future study might improve it even further.

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International Journal of
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